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STABILITY OF GROUP REPRESENTATIONS AND HAAR SPECTRUM

BY

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ABSTRACT. If U and V are commuting unitary representations of locally compact abelian groups S and T, new representations of S (perturbations of U) can be obtained from composition with images of U in V. If most of these representations are equivalent to U, U is said to be V stable. We investigate conditions which, together with stability, ensure that U has (uniform) Haar spectrum. The principal applications are to dynamical systems which possess auxiliary groups with respect to which motion is stable.

0. **Introduction.** The notion of a dynamical system, which is stable with respect to the action of a compact abelian group, was introduced and exploited in [1] to show that under certain conditions the associated group representation has countable Lebesgue spectrum in the orthocomplement of the space of functions invariant under the compact group. In particular this method led to the conclusion that ergodic unipotent affine transformations on nilmanifolds have countable Lebesgue spectrum in the orthocomplement of the eigenfunctions, with a similar conclusion for nilflows; this latter result was first established by Green [2].

The purpose of this paper is to investigate stability in the context of unitary representations of locally compact abelian groups. Our aim is to establish conditions which will ensure that a representation has a Haar spectrum with uniform multiplicity. Apart from the result mentioned above applications of our main theorems yield the well-known results:

[3], [4]. *If U, V satisfy the Weyl commutation relation then U and V have Haar spectrum with uniform multiplicity.*

[5], [2]. *The horocycle flow has Lebesgue spectrum with uniform multiplicity in the orthocomplement of the constant functions.* (Actually, the multiplicity is infinite but our methods will not yield this.)

Let S, T be two locally compact separable abelian groups, and let X, Y be their duals. Let U, V be unitary representations of S, T, in the Hilbert space H, which commute with one another (i.e., $U_sV_t = V_tU_s$ for all $s \in S$, $t \in T$ or, alternatively, U and V are restrictions to S and T of a representation of $S \times T$). In

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this case U is said to be *stable* with respect to V if the representations $s \to U_s$ and $s \to U_s V_{\rho(s)}$ are equivalent for almost all $\rho \in \mathrm{Hom}(S, T)$, the abelian group of continuous homomorphisms from S to T. (The latter group is locally compact when endowed with the compact-open topology, and "almost all" refers to the Haar measure on this group.)

If a unitary operator W_ρ which effects this equivalence $(W_\rho^{-1} U_s W_\rho = U_s V_{\rho(s)})$ can be chosen so that $W_\rho^{-1} V_t W_\rho = V_t$ then U will be called *very stable* with respect to V.

If S, T are locally compact separable abelian groups then we say that there are *sufficiently many homomorphisms* of S into T if for each $(x, y) \in X \times Y$ (X, Y the duals of S, T) with $y \neq 0$ there is a continuous homomorphism $\eta\colon Y \to X$ such that $x = \eta(y)$.

Our main results can be summarized as follows:

Let S, T be locally compact separable abelian groups for which there are sufficiently many homomorphisms of S into T, and let U, V be unitary representations of S, T into the Hilbert space H which commute with one another. Suppose also that 0 is the only V invariant vector.

If U is stable with respect to V then the maximal spectral type of U is Haar measure on X.

If U is faithful and stable with respect to itself then U has Haar maximal spectral type with uniform multiplicity.

If U is very stable with respect to V then U has Haar maximal spectral type with uniform multiplicity.

After proving these results we investigate pairs S, T for which there are sufficiently many homomorphisms. Our final section is devoted to applications.

1. **Preliminaries.** If S is a locally compact abelian group and X is its dual, the natural scalar product of $s \in S$ and $x \in X$ is denoted by $\langle s, x\rangle$. For $x \in X$, δ_x represents the unit mass measure at the point x. The convolution of two positive Borel measures μ, ν on X is denoted $\mu * \nu$. π_X denotes Haar measure on X.

If X, Y are complete separable metric spaces and $\{\mu_y\colon y \in Y\}$ is a family of positive Borel measures on X such that for each Borel subset E of X, $y \to \mu_y(E)$ is Borel measurable, then for any positive Borel measure ν on Y we obtain a measure on X, $E \to \int_Y \mu_y(E)\, d\nu(y)$ which we denote by $\int_Y \mu_y \, d\nu(y)$.

For two measures μ, ν we say $\mu \ll \nu$ (μ is absolutely continuous with respect to ν) if $\nu E = 0$ implies $\mu E = 0$. If $\mu \ll \nu$ and $\nu \ll \mu$ we say that μ is equivalent to ν, $\mu \sim \nu$. If μ is supported by a set on which ν vanishes then we say that μ and ν are disjoint or mutually singular $\mu \perp \nu$.

If S is a locally compact abelian separable group and X is its dual, and if μ is a Borel measure on X, $L(\mu)$ denotes the standard unitary representation of S in $L^2(X, \mu)$ defined by

$$(L(\mu)_s f)(x) = \langle s, x\rangle f(x).$$

In fact, any locally simple representation is equivalent to such a representation. Basic facts concerning representations and multiplicity may be found in [4], [6], and [7].

If U^k are representations of S in Hilbert spaces H^k then $\bigoplus \Sigma_{k=1}^{\infty} U^k$ denotes the direct sum representation in $\bigoplus \Sigma_{k=1}^{\infty} H^k$ and kU denotes the direct sum of k copies of the representation U.

The canonical representation theorem states that each unitary representation of S is equivalent to $\bigoplus \Sigma_{k=1}^{\infty} kL(\mu_k)$ where each $L(\mu_k)$ is the standard representation in $L^2(X, \mu_k)$ and where the measures μ_k are mutually singular (some may be zero). The measure $\mu = \Sigma_{k=1}^{\infty} \mu_k$ or any measure equivalent to it is called the *maximal spectral type of* U.

The maximal spectral type μ is characterized by the following two properties:

(i) For each $v \in H$, $\tilde{v} \ll \mu$.

(ii) For each finite Borel measure $\nu \ll \mu$ there exists $v \in H$ with $\tilde{v} \sim \nu$.

The measures $\tilde{v}$ here, are given by Bochner's theorem:

$$[U_s v, v] = \int_X \langle s, x\rangle d\tilde{v}(x).$$

We shall also need Stone's theorem: There is a unique projection valued measure P defined on the Borel subsets of X such that $U_s = \int_X \langle s, x\rangle \, dP$.

If U is a unitary representation of the direct product $S \times T$ of two locally compact separable abelian groups S, T and if $p : S \times T \to S$ is the natural projection, then the maximal spectral type of $U|S$ is the image by p of the maximal spectral type of U.

If $\{A_k : k = 1, 2, \cdots, \infty\}$ is a partition of X such that μ_k vanishes outside A_k then $\Sigma_{k=1}^{\infty} k\chi_{A_k}$ is called *the multiplicity function* of U. Of course, the multiplicity function is uniquely defined only up to sets of measure zero with respect to μ. The maximal spectral type μ and the multiplicity function determine U up to equivalence.

If U^k $(k = 1, 2, \cdots)$ are unitary representations with maximal spectral types μ_k, then $\bigoplus \Sigma_{k=1}^{\infty} U^k$ has maximal spectral type $\Sigma_{k=1}^{\infty} \mu_k$. Moreover, it is possible to choose for each k a version f_k of the multiplicity function of U^k such that $\Sigma_{k=1}^{\infty} f_k$ is the multiplicity function of $\bigoplus \Sigma_{k=1}^{\infty} U^k$.

U is said to have uniform multiplicity (equal to n, $n = 1, 2, \cdots, \infty$) if its multiplicity function is constant (equal to n) a.e. $[\mu]$, i.e., if $\mu_k \equiv 0$ for $k \neq n$.

2. Main theorems.

Theorem 1. *Let S, T be locally compact separable abelian groups for which there are sufficiently many homomorphisms of S into T. If U, V are commuting unitary representations of S, T in the Hilbert space H such that U is stable with respect to V and such that* 0 *is the only V invariant vector then the maximal spectral type of U is Haar measure.*

Proof. Let X, Y denote the duals of S, T and let R be the representation of $S \times T$ given by $(s, t) \longrightarrow U_s V_t$. There is a null subset N (with respect to Haar measure) of $\operatorname{Hom}(S, T)$ such that if $\rho \in \operatorname{Hom}(S, T) - N$ then U_s is equivalent to $U_s V_{\rho(s)}$.

To each $\rho \in \operatorname{Hom}(S, T)$ we associate $\hat{\rho} \in \operatorname{Hom}(Y, X)$ uniquely defined by

$$\langle \rho(s), y \rangle = \langle s, \hat{\rho}(y) \rangle, \qquad s \in S, \; y \in Y.$$

Let $F_{\hat{\rho}}\colon X \times Y \longrightarrow X \times Y$ be defined by

$$F_{\hat{\rho}}(x, y) = (x + \hat{\rho}(y), y), \qquad (x, y) \in X \times Y.$$

If Q is any unitary representation of $S \times T$, ρQ denotes the representation $(s, t) \longrightarrow Q_{(s, \rho(s)+t)}$. In particular if $L(\mu)$ is the standard representation of $S \times T$ on $L^2(X \times Y, \mu)$ then

$$A \rho L(\mu) = L(F_{\hat{\rho}}(\mu))A, \qquad \rho \in \operatorname{Hom}(S, T),$$

where A is the isometry defined by

$$Af = f \circ F_{\hat{\rho}}^{-1}, \qquad f \in L^2(X \times Y, \mu).$$

Hence if $R \simeq \bigoplus \sum_{k=1}^{\infty} kL(\nu_k)$, then

$$\rho R \simeq \bigoplus \sum_{k=1}^{\infty} k \rho L(\nu_k) \simeq \bigoplus \sum_{k=1}^{\infty} kL(F_{\hat{\rho}}(\nu_k)).$$

By assumption (stability) we have $\rho R | S \simeq U$ if $\rho \in \operatorname{Hom}(S, T) - N$ and certainly $\rho R | S \simeq \bigoplus \sum_{k=1}^{\infty} k\{LF_{\hat{\rho}}(\nu_k)|S\}$ therefore $U \simeq \bigoplus \sum_{k=1}^{\infty} k(L(F_{\hat{\rho}}(\nu_k))|S)$ for almost all $\hat{\rho} \in \operatorname{Hom}(Y, X) \simeq \operatorname{Hom}(S, T)$. In particular, the maximal spectral type μ of U is given by

$$\mu = \sum_{k=1}^{\infty} p \circ F_{\hat{\rho}}(\nu_k) = p \circ F_{\hat{\rho}}(\alpha) \tag{1.1}$$

for almost all $\hat{\rho} \in \operatorname{Hom}(Y, X)$ where $\alpha = \sum_{k=1}^{\infty} \nu_k$.

We can decompose the measure α as follows:

$$\alpha = \int_Y \beta_y \, d\theta(y) \tag{1.2}$$

where β_y is supported by $X \times \{y\}$ for $y \in Y$ and θ is the image of α by the projection from $X \times Y$ to Y.

Let $\epsilon_y = p(\beta_y)$ $(\equiv \beta_y \circ p^{-1})$, then a trivial computation shows that

$$p \circ F_{\hat{\rho}}(\alpha) = \int_Y \delta_{\hat{\rho}(y)} * \epsilon_y \, d\theta(y). \tag{1.3}$$

Let π be a bounded positive measure on Hom (Y, X) equivalent to Haar measure. From (1.1) and (1.3) we get

$$\begin{aligned} \mu &\sim \int_{\mathrm{Hom}(Y,X)} p \circ F_{\hat{\rho}}(\alpha)\, d\pi(\hat{\rho}) \\ &= \int_Y \left[\int_{\mathrm{Hom}(Y,X)} \delta_{\hat{\rho}(y)}\, d\pi(\hat{\rho}) \right] * \epsilon_y\, d\theta(y). \end{aligned} \tag{1.4}$$

But $\int_{\mathrm{Hom}(Y,X)} \delta_{\hat{\rho}(y)} d\pi(\hat{\rho})$ is simply the image of π by the homomorphism from Hom (Y, X) to X given by $\hat{\rho} \longrightarrow \hat{\rho}(y)$. This map is assumed to be onto (sufficiently many homomorphisms) if $y \neq 0$; hence

$$\int_{\mathrm{Hom}(Y,X)} \delta_{\hat{\rho}(y)}\, d\pi(\hat{\rho}) \sim \pi_X \quad \text{(Haar measure on } X\text{)}, \ y \neq 0. \tag{1.5}$$

On the other hand the maximal spectral type of $V = R|T$ is the projection of the maximal spectral type of R, which is by construction θ. Since V has no invariant vector other than 0, we must have $\theta(\{0\}) = 0$. Finally from (1.4) and (1.5) we have

$$\mu \sim \int_Y \pi_X * \epsilon_y\, d\theta(y) \sim \int_Y \pi_X\, d\theta(y) \sim \pi_X,$$

and the proof of the theorem is complete.

Corollary. *Under the hypotheses of the theorem, and with the further assumptions that S is discrete and T is compact, U has Haar maximal spectral type with uniform multiplicity.*

Proof. With the same notation as in Theorem 1, the dual Y of T is discrete and countable, and the dual X of S is compact. As before, let $R \simeq \bigoplus \Sigma_{k=1}^{\infty} kL(\nu_k)$ and let $\nu_{k,y}$ be the measures defined on X by $\nu_k | X \times \{y\} \equiv \nu_{k,y} \otimes \delta_y$. Obviously $\nu_k = \Sigma_{y \in Y} \nu_{k,y} \otimes \delta_y$ and $F_{\hat{\rho}}(\nu_k) = \Sigma_{y \in Y} (\delta_{\hat{\rho}(y)} * \nu_{k,y}) \otimes \delta_y$.

If m is any bounded measure on X we have $L(m \otimes \delta_y)|S \simeq L(m)$. Moreover

$$L(F_{\hat{\rho}}(\nu_k))|S \simeq \sum_{y \in Y} L(\delta_{\hat{\rho}(y)} * \nu_{k,y})$$

and we get, for almost all $\hat{\rho} \in$ Hom (Y, X),

$$U \simeq \bigoplus \sum_{k=1}^{\infty} \sum_{y \in Y} kL(\delta_{\hat{\rho}(y)} * \nu_{k,y}).$$

Since $\Sigma_{k=1}^{\infty} \nu_k$ is the maximal spectral type of R and since $R|S = U$ we have $p(\Sigma_{k=1}^{\infty} \nu_k) \sim$ maximal spectral type of $U = \pi_X$. In particular

$$\nu_{k,y} = p(\nu_{k,y} \otimes \delta_y) \ll p(\nu_k) \ll \pi_X$$

for $1 \le k \le \infty$, $y \in Y$. Let f_y denote the multiplicity function of the representation $U^y = \bigoplus \Sigma_{k=1}^{\infty} kL(\nu_{k,y})$. Then the multiplicity function h_y of $\bigoplus \Sigma_{k=1} kL(\delta_{\hat\rho(y)} * \nu_{k,y})$ is given by $h_y(x) = f(x + \hat\rho(y))$ and since $U \simeq \bigoplus \Sigma_{y \in Y} \Sigma_{k=1}^{\infty} kL(\delta_{\hat\rho(y)} * \nu_{k,y})$ for almost all $\hat\rho \in \mathrm{Hom}(Y, X)$ we have

$$f(x) = \sum_{y \in Y - \{0\}} h_y(x) = \sum_{y \in Y-\{0\}} f_y(x + \hat\rho(y))$$

for π_X a.a. $x \in X$. By Fubini's theorem we see that, for π_X a.a. $x \in X$,

$$f(x) = \sum_{y \in Y - \{0\}} f(x + \hat\rho(y)) \tag{1.6}$$

holds for almost all $\hat\rho$ in $\mathrm{Hom}(Y, X)$.

Assume now that S is discrete, i.e., X is compact. Since Y is discrete and countable, then $\mathrm{Hom}(Y, X)$ is compact. Let its Haar measure π be normalized. We may integrate (1.6) with respect to π to obtain

$$f(x) = \sum_{y \in Y - \{0\}} \int_{\mathrm{Hom}(Y,X)} f_y(x + \hat\rho(y))\, d\pi(\hat\rho)$$

for π_X a.a. $x \in X$. On the other hand, the homomorphism $\hat\rho \longrightarrow \hat\rho(y)$ of $\mathrm{Hom}(Y, X)$ to X is onto if $y \neq 0$ and the image of π by this map is π_X. Hence

$$\int_{\mathrm{Hom}(Y,X)} f_y(x + \hat\rho(y))\, d\pi(\hat\rho) = \int_X f_y(x + x')\, d\pi_X(x') = \int_X f_y(x')\, d\pi_X(x')$$

which is independent of x. In other words, $f(x) = \text{constant}\ [\pi_X]$.

Lemma. *Let S, T be locally compact separable abelian groups and let U, V be commuting unitary representations. Assume V to be faithful. If W_ρ is a unitary operator with $W_\rho^{-1} U_s W_\rho = U_s V_{\rho(s)}$ and $W_\rho^{-1} V W_\rho = V$ (i.e., $\{W_\rho^{-1} V_t W_\rho: t \in T\} \equiv \{V_t: t \in T\}$) where ρ is a continuous homomorphism of S into T then $W_\rho H_k = H_k$ where H_k is the subspace on which $R = U \times V$ has uniform multiplicity k.*

Proof. Evidently, $W_\rho^{-1} V_t W_\rho = V_{\alpha(t)}$ where α is a continuous automorphism of T depending on ρ. Continuity follows from the equation, $[V_{\alpha(t)} v, v] = [V_t W_\rho v, W_\rho v]$, the faithfulness of V and an application of the closed graph theorem.

Let H_k be the subspace of H such that $R = U \times V$ restricted to H_k has multiplicity k. Then there exist $v_1, v_2, \cdots, v_k$ generating orthogonal cycles C_1, $C_2, \cdots, C_k$ (C_i = closed linear span of $\{U_s V_t v_i: (s, t) \in S \times T\}$) and $\tilde v_1 = \tilde v_2 = \cdots = \tilde v_k$ where $[U_s V_t v_i, v_i] = \int_{X \times Y} \langle s, x\rangle \langle t, y\rangle\, d\tilde v_i$.

Since

$$\begin{aligned}
[U_s V_t W_\rho v_i, W_\rho v_j] &= [U_s V_{\rho(s)+\alpha(t)} v_i, v_j] = 0, && i \neq j, \\
&= \int_{X \times Y} \langle x, s\rangle \langle y, \rho(s) + \alpha(t)\rangle\, d\tilde v_i, && i = j, \\
&= \int_{X \times Y} \langle x, s\rangle \langle y, t\rangle\, dF_{\hat\rho}(\tilde v_i)
\end{aligned}$$

where $F_{\hat{\rho}}(x, y) = (x + \hat{\rho}(y), \hat{\alpha}(y))$ we see that $W_\rho v_1, W_\rho v_2, \cdots, W_\rho v_k$ also generate k orthogonal cycles with the same spectral type. Hence $W_\rho H_k \subset \bigoplus_{n=k}^{\infty} H_n$, $k = 1, 2, \cdots$. The same argument may of course be applied to obtain $W_\rho^{-1} H_k \subset \bigoplus_{n=k}^{\infty} H_n$, $k = 1, 2, \cdots$, since α is an automorphism. We thus have $W_\rho H_k = H_k$.

Theorem 2. *Let S be a separable locally compact abelian group with sufficiently many selfhomomorphisms. Let U be a faithful unitary representation of S with no invariant vectors other than 0. If U is selfstable (i.e., U is stable with respect to U) then the maximal spectral type of U is Haar measure and U has uniform multiplicity.*

Proof. Let R be the representation of $S \times S$ given by $(s, t) \longrightarrow U_{s+t}$. By the lemma we may suppose that R has uniform multiplicity say k. By Theorem 1, U has Haar measure as maximal spectral type. Hence there are vectors $v_1, v_2, \cdots, v_k$ generating orthogonal R cycles (and therefore U cycles) $C_1, \cdots, C_k$ with $[U_{s+t} v_i, v_i] = \int_{X \times X} \langle x, s\rangle \langle y, t\rangle \, d\tilde{v}_i$ where all the $\tilde{v}_i$, $1 \le i \le k$, are identical to the maximal spectral type of R. Hence $[U_s v_i, v_i] = \int_{X \times X} \langle x, s\rangle \, d\tilde{v}_i = \int_X \langle x, s\rangle \, dp\tilde{v}_i$ where p is the projection of $X \times X$ to X. In any case $p\tilde{v}_i \equiv p\tilde{v}_2 \equiv \cdots \equiv p\tilde{v}_k$ and therefore U has uniform multiplicity.

Theorem 3. *Let S, T be locally compact separable abelian groups with sufficiently many homomorphisms from S to T. If U, V are commuting unitary representations such that 0 is the only V invariant vector and if U is very stable with respect to V then U has Haar measure as maximal spectral type and with uniform multiplicity.*

Proof. Let N be a null set in $\mathrm{Hom}(S, T)$ for which $W_\rho^{-1} U_s W_\rho = U_s V_{\rho(s)}$ and $W_\rho^{-1} V_t W_\rho = V_t$ for some unitary operator W_ρ where $\rho \in \mathrm{Hom}(S, T) - N$. If $V_t = \int_Y \langle y, t\rangle \, dQ$ then $W_\rho^{-1} Q W_\rho \equiv Q$ and $U^{-1} Q U \equiv Q$. Hence the hypotheses of the theorem remain valid for the subrepresentations on $Q(M)H$ for each Borel subset M of Y. If there exists a countably infinite partition $M_1, M_2, \cdots$ of Y for which $Q(M_i) \not\equiv 0$ then U has Haar maximal spectral type on each of the spaces $Q(M_i)H$ and therefore U has infinite uniform multiplicity.

The alternative is that Q is concentrated on a finite number of points $y_1, y_2, \cdots, y_k$, where $Q(y_i) \neq 0$, and it is clear that each of the spaces $Q(y_i)H$ is cyclic with respect to U. Since $U|Q(y_i)H$ has Haar maximal spectral type (Theorem 1) we see that U has Haar maximal spectral type with uniform multiplicity k.

3. **Sufficiently many homomorphisms.** We now give a characterization of the pairs S, T to which the main theorems may be applied.

Theorem 4. *Let S, T be two nontrivial separable locally compact abelian groups which are generated by compact neighborhoods of their identities. There are sufficiently many homomorphisms of S into T (i.e., for each $(x, y) \in X \times Y$, $y \neq 0$,* *there is a continuous homomorphism* $\hat{\rho}\colon Y \to X$ *with* $x = \hat{\rho}(y)$) *if and only if* (S, T) *belongs to one of the following types*:

$$S = \mathfrak{R}^p \times \mathbf{Z}^k, \qquad T = \mathfrak{R}^q \times H \tag{1}$$

where p, k, q *are integers* $q \neq 0$ *and* H *is a compact connected abelian group.*

$$S = \mathfrak{R}^p \times \mathbf{Z}^k \times K, \qquad T = H \tag{2}$$

where p, k *are arbitrary integers,* H *is a compact connected abelian group and* K *is a compact abelian group whose dual is divisible.*

(3) S, T *are compact abelian groups all of whose elements are of order* p *where* p *is an arbitrary prime integer.*

Proof. We shall say that the pair (X, Y) of locally compact separable abelian groups has the property (P) if for every pair $(x, y) \in (X, Y)$ with $y \neq 0$ there is a continuous homomorphism $\phi\colon Y \to X$ such that $\phi(y) = x$, or equivalently if there are sufficiently many homomorphisms of S into T where S, T are the duals of X, Y respectively. We call $\mathrm{Tor}(X)$, $\mathrm{Tor}(Y)$ the torsion groups of X and Y.

(1) Assume first that $\mathrm{Tor}(Y) \neq 0$. Let $y \neq 0$ be of finite order k in Y; property (P) for (X, Y) implies that all elements x of X satisfy $k \cdot x = 0$. Let p be the smallest integer such that $p \cdot x = 0$ for all $x \in X$. Let $n > 1$ be a nontrivial divisor of p so that $p = nm$ with $m < p$. If there were in Y a $y \neq 0$ such that $n \cdot y \neq 0$ then by property (P), each $x \in X$ would have an nth root x_n and thus $m \cdot x = m \cdot (n \cdot x_n) = p \cdot x_n = 0$, $x \in X$, which is impossible since $m < p$. So we must have $n \cdot y = 0$ for all $y \in Y$. This in turn implies $n \cdot x = 0$ for all $x \in X$ by property (P), which imposes $n = p$. Thus p is a prime integer and every element of X and Y is of order p.

Since S, T are generated by compact neighborhoods of 0 we have [8, p. 110]

$$\begin{aligned} S &= \mathfrak{R}^n \times \mathbf{Z}^k \times K, & X &= \mathfrak{R}^n \times C^k \times \hat{K}, \\ T &= \mathfrak{R}^q \times \mathbf{Z}^r \times H, & Y &= \mathfrak{R}^q \times C^r \times \hat{H}, \end{aligned} \tag{3.1}$$

where C is the one-dimensional torus, and K, H are separable compact abelian groups, with dual groups $\hat{K}$, $\hat{H}$. Clearly we must have $X = \hat{K}$, $Y = \hat{H}$, where $\hat{K}$ and $\hat{H}$ are countable discrete groups all of whose elements have order p. We show that, in this situation property (P) holds:

Any subgroup of Y generated by a finite number of elements is finite and hence has the form $[\mathbf{Z}/p\mathbf{Z}]^m$ for some m [9, Theorem 3.3.1, p. 40]. Since p is a prime integer, $\mathbf{Z}/p\mathbf{Z}$ is a field, and every $\mathbf{Z}/p\mathbf{Z}$-module is $\mathbf{Z}/p\mathbf{Z}$-divisible, and hence injective in the category of $\mathbf{Z}/p\mathbf{Z}$ modules [10, Theorem 4.2., p. 11]. Then property (P) holds trivially for $(X, [\mathbf{Z}/p\mathbf{Z}]^m)$ and if G is any countable inductive limit of $\mathbf{Z}/p\mathbf{Z}$-modules of the type $[\mathbf{Z}/p\mathbf{Z}]^m$, the fact that X is injective shows readily that (X, G) has property (P). In particular (X, Y) has property (P). This achieves the proof of part (3) of Theorem 4.

(2) Assume now Tor $Y = 0$, and that the connected component Y_0 of 0 in Y is not trivial. Then by formula (3.1), $Y = \mathfrak{R}^q \times \hat{H}$ where $q \neq 0$ and Tor $\hat{H} = 0$, i.e., $T = \mathfrak{R}^q \times H$ where $q \neq 0$, H compact connected. The image of $Y_0 = \mathfrak{R}^q$ by any continuous homomorphism from Y to X must be contained in the connected component X_0 of 0 in X. If (X, Y) has property (P) we must then have $X_0 = X$, i.e., $X = \mathfrak{R}^n \times C^k$ and $S = \mathfrak{R}^n \times \mathbf{Z}^k$. We now show that $(X = \mathfrak{R}^n \times C^k,\ Y = \mathfrak{R}^q \times H)$ with Tor $\hat{H} = 0$ has property (P). Since $(\mathfrak{R}^n \times C^k, \mathfrak{R}^q)$ trivially has property (P) it is enough to check that $(X, \hat{H})$ has property (P). But X is divisible as a $\mathbf{Z}$-module and hence injective in the category of $\mathbf{Z}$-modules [10, Proposition 5.1, p. 134]. Then property (P) holds for $(X, \mathbf{Z}^n)$ since X is injective for (X, G) where G is any countable inductive limit of groups of the form $\mathbf{Z}^m$. But $\hat{H}$ is discrete, countable, and torsion free; then [8, p. 110] $\hat{H}$ is such an inductive limit and property (P) holds for $(X, \hat{H})$ which achieves the proof of part (1) of Theorem 4.

(3) Assume now that Tor $Y = 0$ and $Y_0 = 0$ then, by the formula (3.1), $Y = \hat{H}$ with Tor $\hat{H} = 0$ so that $T = H$ is compact connected abelian. Fix $y \neq 0$ in Y. Since $m \cdot y \neq 0$ for all integers m, we see that if (X, Y) has property (P), every element of X must have an mth root, for all $m \in \mathbf{Z}$. Thus X is divisible and, by formula (3.1), $X = \mathfrak{R}^n \times C^k \times \hat{K}$ where $\hat{K}$ is discrete divisible. As shown in the second part of the proof, the pair $(X, \hat{H})$ with X divisible, $\hat{H}$ discrete, countable and torsion free, has the property (P). This achieves the proof of part (2) in Theorem 4.

It is now easy to describe precisely the scope of some of the theorems given in the previous sections.

In the Corollary to Theorem 1 (S discrete, T compact) we must have either ($S = \mathbf{Z}^k$, T compact connected) or ($S = [\mathbf{Z}/p\mathbf{Z}]^k$, T compact all elements of order p) where p is a prime.

In Theorem 2 (S with enough selfhomomorphisms) either $S = \mathfrak{R}^q$ or S is a compact connected group with divisible dual, or S is a compact group all elements of which have order p.

4. **Applications.** As mentioned in the introduction one of the principal applications of the method expounded here is to the spectral analysis of unipotent affine transformations on nilmanifolds and of nilflows. Since this was carried out in [1] we shall not give details here.

If S is a locally compact separable abelian group and T is its dual we say that the unitary representations U, V of S, T satisfy the Weyl commutation relation if $U_s V_t = \langle s, t \rangle V_t U_s$. Let C be the one dimensional torus and let W be the representation $W_k v = kv$ for $k \in C$. Then U, W commute and U is very stable with respect to W since $V_t U_s W_{\langle t,s \rangle} = U_s V_t$ for each homomorphism t of S into C.

Theorem 4 shows immediately that there are enough homomorphisms from S to C if and only if the dual T of S is divisible, which we now assume. The hypotheses of Theorem 3 are then satisfied and

U has Haar maximal spectral type with uniform multiplicity.

Our second application concerns the commutation relation between a geodesic and horocycle flow on a surface of constant negative curvature. Let $G = SL(2, \mathfrak{R})$ and let H be a discrete subgroup such that G/H is compact. G acts as a continuous group of homeomorphisms on G/H and there is a unique normalized Borel measure m on G/H which is preserved by the action of G. If

$$S = \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix} = g_s \quad \text{and} \quad T = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = h_t$$

then the action of S on G/H is called the geodesic flow and the action of T on G/H is called the horocycle flow and each action induces a representation of S and T in $L^2(G/H, m)$. We shall assume the known fact that the horocycle flow is ergodic or, equivalently, that the representation $(V_t f)(xH) = f(h_t xH)$ has no invariant vector in H_0 the orthocomplement of the constant functions in $L^2(G/H, m)$. The representation U of S on H_0 is given by $(U_s f)(xH) = f(g_s xH)$.

Moreover, $U_s V_t = V_{e^{-2s}.t} U_s$ and for any representation induced by a measure preserving flow we always have $WV_t = V_{-t}W$, for some unitary operator W. These two relations show that $V_t \simeq V_{t+\rho(t)}$ where ρ is any homomorphism of T to itself other than $t \longrightarrow -t$. In other words V is selfstable and since T has sufficiently many selfhomomorphisms Theorem 2 applies and we see that *the horocycle flow has Lebesgue spectrum with uniform multiplicity.*

REFERENCES

1. W. Parry, *Spectral analysis of G-extensions of dynamical systems*, Topology 9 (1970), 217–224. MR 41 #6194.

2. L. Auslander, L. Green and F. Hahn, *Flows on homogeneous spaces*, Ann. of Math. Studies, no. 53, Princeton Univ. Press, Princeton, N. J., 1963. MR 29 #4841.

3. H. Weyl, *Gruppentheorie und Quantenmechanik*, Hirzel, Leipzig, 1928; reprint, Dover, New York, 1950.

4. G. W. Mackey, *The theory of group representations*, Lecture Notes (Summer, 1955), 3 vols., University of Chicago, Dept. of Math., Chicago, Ill., 1955. MR 19, 117.

5. O. S. Parasyuk, *Flows of horocycles on surfaces of constant negative curvature*, Uspehi Mat. Nauk 8 (1953), no. 3 (55), 125–126. (Russian) MR 15, 442.

6. A. I. Plessner and V. A. Rohlin, *Spectral theory of linear operators*. II, Uspehi Mat. Nauk 1 (1946), no. 1 (11), 71–191; English transl., Amer. Math. Soc. Transl. (1) 62 (1967), 29–175. MR 9, 43.

7. P. R. Halmos, *Introduction to Hilbert space and the theory of spectral multiplicity*, Chelsea, New York, 1951. MR 13, 563.

8. A. Weil, *Intégration dans les groupes topologiques et ses applications*, 2nd ed., Actualités Sci. Indust., no 869, Hermann, Paris 1951.

9. M. Hall, Jr., *The theory of groups*, Macmillan, New York, 1959. MR 21 #1996.

10. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N. J., 1956. MR 17, 1040.

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